JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 97, 214–218 (1983)

Operator Norms on $L(E)$

JEN-CHUNG CHUAN*

Institute of Mathematics,
National Tsing Hua University, Hsinchu, Taiwan 300, Republic of China

AND

KOK-KEONG TAN†

Department of Mathematics, Statistics and Computing Science,
Dalhousie University, Halifax, Nova Scotia B3H 4H8, Canada

Submitted by C. L. Dolph

Let $(E, \| \|)$ be a Banach space and $L(E)$ be the Banach algebra of all bounded linear operators on E. Two characterizations are obtained on equivalent norms on $L(E)$ being operator norms. As an application, it is shown that if $U, V \in L(E)$ satisfying $VU = I$ and $\|UV\| = 1$, define $\|T\|^* = \|UTV\|$ for all $T \in L(E)$, then $\| \|^*$ is an operator norm on $L(E)$.

1. PRELIMINARIES

Throughout this paper, (E, q) denotes a Banach space and $L(E)$ the algebra of all bounded linear operators on E. We observe that $(L(E), q^*)$ is a unital Banach algebra, where q^* is the norm on $L(E)$ defined by

$$q^*(T) = \sup\{q(Tx) : q(x) \leqslant 1\} \qquad \text{for all} \quad T \in L(E).$$

q^* is called the norm induced by q.

DEFINITION 1.1. A norm p on $L(E)$ is an operator norm on $L(E)$ if there exists a norm s on E equivalent to q such that $p = s^*$, where s^* is the norm on $L(E)$ induced by s.

In this paper, we shall characterize all norms on $L(E)$ that are operator norms. First we observe that if s is any norm on E equivalent to q, then (i) s^* is an operator norm on $L(E)$, (ii) $(L(E)), s^*)$ is a unital Banach algebra, and (iii) s^* is equivalent to q^*. Let N be the set of all unital Banach algebra

* Partially supported by National Science Council, Taiwan, Republic of China.
† Partially supported by NSERC of Canada under Grant A-8096.

0022-247X/83 $3.00

norms on $L(E)$. Since $(L(E), q^*)$ is semi-simple, a theorem of Johnson [1, p. 130] implies the following.

THEOREM 1.2. *If $p \in N$, then p is equivalent to q^*.*

Thus to characterize all norms on $L(E)$ that are operator norms, we need only characterize all elements in N that are operator norms.

2. MAIN RESULT

Fix a nonzero continuous linear functional f on E throughout the rest of this paper.

Notation 2.1. For each $p \in N$, p_* is defined as

$$p_*(x) = p(x \otimes f) \qquad \text{for all} \quad x \in E,$$

where $x \otimes f$ stands for the rank one operator given by

$$(x \otimes f)(y) = f(y)\, x \qquad \text{for all} \quad y \in E.$$

LEMMA 2.2. *If $p \in N$, then p_* is a norm on E equivalent to q.*

Proof. Let $p \in N$. It is easy to see that p_* is a norm on E. By Theorem 1.2, p is equivalent to q^*, so that there are positive constants m and M such that

$$mp(T) \leqslant q^*(T) \leqslant Mp(T) \qquad \text{for all} \quad T \in L(E).$$

Letting $q(f) = \sup\{|f(y)| : q(y) \leqslant 1\}$, the equality $q^*(x \otimes f) = q(f)\, q(x)$ now gives

$$mp_*(x) = mp(x \otimes f) \leqslant q(f)\, q(x) \leqslant Mp(x \otimes f) = Mp_*(x);$$

thus p_* and q are equivalent.

LEMMA 2.3. *If $p \in N$, then $p_*{}^* \leqslant p$.*

Proof. If $T \in L(E)$, we have, for all $x \in E$,

$$p_*(Tx) = p((Tx) \otimes f) = p(T(x \otimes f)) \leqslant p(T)\, p(x \otimes f) = p(T)\, p_*(x);$$

therefore, $p_*{}^*(T) \leqslant p(T)$ for all $T \in L(E)$ so that $p_*{}^* \leqslant p$.

Notation 2.4. Let E' denote the continuous dual of E. If s is any norm on E equivalent to q and $g \in E'$, let

$$s(g) = \sup\{|g(x)| : s(x) \leqslant 1\}.$$

By a direct computation, we have the following

LEMMA 2.5. *If $x \in E$, $g \in E'$, and $T \in L(E)$, we have*

$$T(x \otimes g)\, T(x \otimes g) = g(Tx)\, T(x \otimes g).$$

LEMMA 2.6. *If s is a norm on E equivalent to q, then $s^*{}_*{}^* = s^*$. In particular, if $p \in N$ is an operator norm on $L(E)$, then $p_*{}^* = p$.*

Proof. If $T \in L(E)$, we have

$$\begin{aligned} s^*{}_*{}^*(T) &= \sup\{s^*{}_*(Ty) \mid s^*{}_*(y) \leqslant 1\} \\ &= \sup\{s^*(Ty \otimes f) \mid s^*(y \otimes f) \leqslant 1\} \\ &= \sup\{s(f)\, s(Ty) \mid s(f)\, s(y) \leqslant 1\} \\ &= s^*(T), \end{aligned}$$

thus $s^*{}_*{}^* = s^*$.

By applying the Hahn–Banach extension theorem we have

LEMMA 2.7. *If s is a norm on E equivalent to q, then*

$$s^*(T) = \sup\{|g(Tx)| : s(x) \leqslant 1, s(g) \leqslant 1, x \in E, g \in E'\},$$

for all $T \in L(E)$. In particular, if $p \in N$ is an operator norm on $L(E)$, we have

$$p(T) = \sup\{|g(Tx)| : p_*(x) \leqslant 1, p_*(g) \leqslant 1, x \in E, g \in E'\},$$

for all $T \in L(E)$.

THEOREM 2.8. *Let $p \in N$. Then the following conditions are equivalent:*

(1) *p is an operator norm on $L(E)$;*

(2) *$p = p_*{}^*$;*

(3) *p is minimal in N.*

Proof. By Lemma 2.6, we have (1) $\Rightarrow$ (2) and by Lemma 2.2, we have (2) $\Rightarrow$ (1).

(2) $\Rightarrow$ (3). Note that p_* is a norm on E equivalent to q, by Lemma 2.2. Let $p_1 \in N$ be such that $p_1 \leqslant p$. Let $T \in L(E)$ be given. Then $p_1(T) \leqslant p(T)$.

On the other hand, let $x \in E$ and $g \in E'$ be such that $p_*(x) \leqslant 1$, and $p_*(g) \leqslant 1$. By Lemma 2.5, we have

$$T(x \otimes g)\, T(x \otimes g) = g(Tx)\, T(x \otimes g)$$

so that

$$\begin{aligned} |g(Tx)|\, p_1(T(x \otimes g)) &\leqslant p_1(T(x \otimes g))^2; \\ |g(Tx)| &\leqslant p_1(T(x \otimes g)) \\ &\leqslant p_1(T)\, p_1(x \otimes g) \\ &\leqslant p_1(T)\, p(x \otimes g) \\ &= p_1(T)\, p_*{}^*(x \otimes g) \\ &= p_1(T)\, p_*(x)\, p_*(g) \\ &\leqslant p_1(T). \end{aligned}$$

Thus by Lemma 2.7,

$$\begin{aligned} p(T) &= p_*{}^*(T) \\ &= \sup\{|g(Tx)| : p_*(x) \leqslant 1, p_*(g) \leqslant 1, x \in E, g \in E'\} \\ &\leqslant p_1(T). \end{aligned}$$

Hence $p(T) = p_1(T)$ for all $T \in L(E)$ so that $p = p_1$. Therefore, p is minimal in N.

$(3) \Rightarrow (2)$. By Lemma 2.2, $p_*{}^* \in N$. By Lemma 2.3, $p_*{}^* \leqslant p$. Since p is minimal in N, we must have $p = p_*{}^*$.

We remark that in Theorem 2.7, "$(1) \Leftrightarrow (3)$" generalizes the work of Ljubič on "Operator norms of matrices" (see [3] or [2, p. 242]). Also, if we define, for each $p \in N$ and $f \in E'$, $p_f(x) = p(x \otimes f)$ for all $x \in E$, Lemma 2.5 actually shows that $P_f^* = p_g^*$ for all $p \in N$ and all nonzero $f, g \in E'$. Thus, to determine whether a given $p \in N$ is an operator norm on $L(E)$, condition (2) is much more practical than condition (3) in Theorem 2.7.

3. An Application

THEOREM 3.1. *Let U, $V \in L(E)$ be such that $VU = I$ and $q^*(UV) = 1$. Define*

$$p(T) = q^*(UTV) \qquad \textit{for all} \quad T \in L(E),$$

then $p \in N$ and p is an operator norm on $L(E)$.

Proof. It is easy to see that p is a norm on $L(E)$. Since $VU=I$, p is an algebra norm; also since $q^*(UV)=1$, p is unital. Let $(T_n)_{n=1}^{\infty}$ be a sequence in $L(E)$ such that $p(T_n-T_m)\to 0$ as $n,m\to\infty$, then $q^*(T_n-T_m)=q^*(VU(T_n-T_m)\,VU)\leqslant q^*(V)\,p(T_n-T_m)\,q^*(U)\to 0$ as n, $m\to\infty$; since $(L(E),q^*)$ is complete, there exists a $T\in L(E)$ such that $q^*(T_n-T)\to 0$ as $n\to\infty$. Thus $p(T_n-T)=q^*(U(T_n-T)\,V)\leqslant q^*(U)\,q^*(T_n-T)\,q^*(V)\to 0$ as $n\to\infty$. Hence $(L(E),p)$ is a unital Banach algebra so that $p\in N$.

Next, let V' be the adjoint of V, then for each $x\in E$, we have

$$\begin{aligned}p_*(x)&=p(x\otimes f)=q^*(U(x\otimes f)\,V)=q^*((Ux)\otimes(V'f))\\&=q(Ux)\,q(V'f);\end{aligned}$$

consequently, for each $T\in L(E)$,

$$\begin{aligned}p_*{}^*(T)&=\sup\{p_*(Tx)\mid p_*(x)\leqslant 1\}\\&=\sup\{q(UTx)\,q(V'f)\mid q(Ux)\,q(V'f)\leqslant 1\}\\&=\sup\{q(UTx)\mid q(Ux)\leqslant 1\}\\&=\sup\{q(UTVy)\mid q(UVy)\leqslant 1\}\qquad\text{(since V is surjective)}\\&\geqslant\sup\{q(UTVy)\mid q(y)\leqslant 1\}\\&=q^*(UTV)\\&=p(T);\end{aligned}$$

thus, $p_*{}^*\geqslant p$. By Lemma 2.3, $p_*{}^*\leqslant p$. Therefore, $p_*{}^*=p$. By Theorem 2.8, p is an operator norm on $L(E)$.

We note that in the above theorem, if $q(U)=q(V)=1$, then in fact $p=q^*$.

COROLLARY 3.2. *Let $U\in L(E)$ be invertible. Define*

$$p(T)=q^*(UTU^{-1})\qquad\textit{for all}\quad T\in L(E).$$

Then $p\in N$ and p is an operator norm on $L(E)$.

REFERENCES

1. F. F. BONSALL AND J. DUNCAN, "Complete Normed Algebras," Springer-Verlag, Berlin/Heidelberg/New York, 1973.
2. I. M. GLAZMAN AND JU. I. LJUBIČ, "Finite-Dimensional Linear Analysis," MIT Press, Cambridge, Mass., 1974.
3. JU. I. LJUBIČ, Operator norms of matrices, *Usp. Mat. Nauk.* **18**, No. 4 (1963), 161–164.